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A note on the static stability of an elastico-viscous fluid

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A recent analysis by Gupta (1967) suggests that a layer of elastico-viscous fluid at rest between parallel plane boundaries may be in unstable equilibrium. This surprising result is attributable to the inadequacy of the constitutive equation adopted by Gupta as the basis for his analysis. An alternative constitutive relation, which takes account of the entire strain-history of the motion, leads to the more reasonable result that the equilibrium is stable whenever the fluid has a 'fading memory'.

1. Introduction

Gupta (1967) has recently examined the stability of a film of elastico-viscous fluid flowing down an inclined plane. In his analysis, Gupta employed the Rivlin–Ericksen constitutive equation for a second-order incompressible fluid (see, for example, Markovitz & Coleman 1964). In addition to solving the stability problem for surface disturbances, he found that 'shear waves' were unstable at small Reynolds numbers and at small wave-numbers, although Yih (1963) has shown that the latter disturbances are highly damped in ordinary Newtonian fluids.

In the approximate analysis for 'shear waves' at low Reynolds numbers, the Rivlin–Ericksen constitutive equation simplifies to the linear form

$$S_{ij}+p\delta_{ij} = \left(\eta_0+\gamma\frac{\partial}{\partial t}\right)e_{ij}, \tag{1.1}$$

where S_{ij} is the stress tensor, e_{ij} is the rate-of-strain tensor, δ_{ij} is the Kronecker delta, p is an indeterminate pressure and η_0, γ are material constants. For a two-dimensional periodic disturbance of (prescribed) wave-number k and x, t-dependence of the form $\exp(ikx+\sigma t)$, the resultant eigenvalue problem for 'shear waves' is specified by the equation

$$(D^2-\alpha^2)(D^2+\lambda^2)\phi = 0,$$

and the boundary conditions

$$\phi = D\phi = 0, \quad y = \pm 1.$$

Here, $\phi \equiv \phi(y)$ and $D \equiv d/dy$; α is a dimensionless wave-number equal to kd, where $2d$ is the film thickness, and λ is given by

$$\lambda^2 = -\alpha^2 - R_\sigma/(1+MR_\sigma),$$

where

$$R_\sigma = \rho\sigma d^2/\eta_0, \quad M = \gamma/\rho d^2,$$

and ρ is the fluid density (see § 5 of Gupta's paper).

Provided we accept the constitutive equation (1.1), the above eigenvalue problem is precisely that for a fluid layer at rest between parallel plane boundaries which experiences a small periodic disturbance.

The even solutions of $\phi(y)$ lead to the eigenvalues

$$R_\sigma = -(\lambda_n^2+\alpha^2)/[1+M(\lambda_n^2+\alpha^2)], \tag{1.2}$$

where λ_n $(n = 1, 2, \ldots)$ is one of the infinite number of real roots of the equation

$$\lambda \tan \lambda = -\alpha \tanh \alpha. \tag{1.3}$$

The eigenvalues for the odd solutions may also be shown to satisfy (1.2), where the λ_n are roots of

$$\lambda \cot \lambda = \alpha \coth \alpha \tag{1.4}$$

(note that the degenerate case $\lambda = \alpha = 0$ is not a solution of the eigenvalue problem). Since γ and hence M are known to be negative (see Markovitz & Coleman), equation (1.2) reveals that there must exist positive values of R_σ which correspond to temporally amplified disturbances. (The result for an ordinary Newtonian fluid is recovered on setting M equal to zero: R_σ is then always negative.) We observe that the growth rates predicted by equation (1.2) always satisfy the inequality $\sigma > |\eta_0/\gamma|$, and that, if M approaches $-(\lambda_n^2+\alpha^2)^{-1}$, the growth rate becomes indefinitely large.

This instability was also noted in an earlier paper by Coleman, Duffin & Mizel (1965). Their paper is primarily a contribution to the theory of partial differential equations, but the authors indicated the relevance of this particular result to the theory of second-order fluids. Whereas Gupta concluded that the elastico-viscous layer was likely to be unstable in practice, Coleman, Duffin & Mizel were careful to emphasize that the use of a more general constitutive relation might yield different results. Also, in a review article, Pipkin (1966) has described the apparent instability as 'an interesting absurdity which arises when the ...... approximation is treated as if it were exact'.

The inadequacy of the constitutive equation (1.1), in the present context, is best indicated as follows. For a fluid layer which is at rest apart from a small perturbation with exponential time-dependence, equation (1.1) simplifies to

$$S_{ij}+p\delta_{ij} = (\eta_0+\sigma\gamma)\, e_{ij}. \tag{1.5}$$

Now, the eigenvalues for σ obtained from equations (1.2)–(1.4) all lead to negative values of $(\eta_0+\sigma\gamma)$; and this implies that the deviatoric stresses have the *opposite* sign to the respective strain-rates. Clearly, no real fluid can possess this property: it implies, for example, that the fluid spontaneously releases energy when given an appropriate initial deformation.

There are other simple rheological models that do not have this limitation. For instance, the constitutive equation for Oldroyd's 'liquid B' (see Oldroyd

1950) is found to yield only stable solutions to the same problem. In fact, this constitutive equation has been adopted by Wei Lai (1967) and by Gupta & Rai (1967) to examine the stability of flow down an inclined plane.

2. Improved constitutive relation

A further limitation of the constitutive equation (1.1) is that it takes little account of the strain history of the motion; and this deficiency is shared by Oldroyd's 'liquid B'. Such models are particularly unreliable for liquids with 'long memories' or for high-frequency phenomena. Improvements in this respect are the more complicated constitutive equations of Oldroyd (1950) and of Coleman & Noll (1961). For linearized motion, the latter authors proposed the relation (see also Markovitz & Coleman)

$$S_{ij}+p\delta_{ij} = 2\int_0^\infty m(s)\,[E_{ij}(t-s)-E_{ij}(t)]\,ds, \tag{2.1}$$

where $m(s)$ is a material function of the fluid and $E_{ij}(t)$ is the infinitesimal strain tensor at time t relative to some fixed configuration. Thus, $E_{ij}(t-s), 0 \leqslant s < \infty$, describes the past history of the fluid and $m(s)$ represents the 'memory' of the fluid at time t, of its state at time $t-s$.

If $E_{ij}(t-s)$ is expanded as a Taylor series at time t, the two leading terms on the right-hand side of (2.1) correspond to the right-hand side of (1.1), with

$$\eta_0 = -\int_0^\infty sm(s)\,ds, \quad 2\gamma = \int_0^\infty s^2m(s)\,ds. \qquad (2.2a, b)$$

Therefore, for disturbances with sufficiently large characteristic time-scales, (1.1) and (2.1) are similar. But, if the time-scale associated with the motion is small compared with the time-scale which characterizes the memory of the fluid, the higher-order derivatives of the Taylor-series expansion cannot be neglected. Since the latter time-scale is typically $O(|\gamma/\eta_0|)$, the relationship (1.5) is likely to be a valid approximation to (2.1) only if $|\sigma| \ll |\eta_0/\gamma|$. This observation emphasizes the shortcomings of the analysis outlined in §1, since σ was then always greater than $|\eta_0/\gamma|$. However, it should be noted that the results of Gupta which relate to surface waves are likely to be fairly accurate, in view of their comparatively small frequencies and rates of amplification: it is only the apparent 'shear-wave' instability which must be rejected.

On physical grounds, we may assert that any physically realistic model should possess the property that a layer of fluid, at rest between horizontal plane rigid boundaries, is in stable equilibrium. It is not obvious, *a priori*, that the constitutive equation (2.1) yields this result for all material functions $m(s)$. However, in the following section, it is shown that the equilibrium is indeed stable for all functions $m(s)$ which represent a gradually-fading memory.

3. The stability problem

For a disturbance with time-dependence of the form exp (σt), the constitutive equation (2.1) is

$$S_{ij}+p\delta_{ij} = 2\sigma^{-1}e_{ij}(t)\int_0^\infty m(s)\,(e^{-\sigma s}-1)\,ds.$$

Also, the equations of motion are

$$\rho\dot{v}_i = S_{ij,j} = -\frac{\partial p}{\partial x_i} + 2\sigma^{-1} e_{ij,j}(t) \int_0^\infty m(s)\,(e^{-\sigma s} - 1)\,ds,$$

$$v_{j,j} = 0,$$

where v_i is the fluid velocity and the overdot denotes $\partial/\partial t$.

It follows from the definition of e_{ij} that $2e_{ij,j} = v_{i,jj} = \nabla^2 v_i$. Introducing the vorticity $\zeta_{ij} = v_{i,j} - v_{j,i}$ and eliminating p by cross-differentiation, we obtain

$$\rho\dot{\zeta}_{ij} = \sigma^{-1}\nabla^2\zeta_{ij}\int_0^\infty m(s)\,(e^{-\sigma s} - 1)\,ds.$$

For a two-dimensional disturbance described by a (dimensionless) stream function of the form $\phi(y'/d)\exp(ikx + \sigma t)$, where $y'/d = y$, the above equation reduces to

$$(D^2 - \alpha^2)(D^2 + \lambda_1^2)\,\phi = 0,$$

with

$$\lambda_1^2 = -\alpha^2 - \rho(\sigma d)^2 \Big/ \int_0^\infty m(s)\,(e^{-\sigma s} - 1)\,ds;$$

also, if the velocity fluctuations vanish at $y' = \pm d$, we have

$$\phi = \phi' = 0, \quad y = \pm 1.$$

The eigenvalue problem so specified is identical to that described in §1 on replacing λ by λ_1. The eigenvalues σ are therefore given by the equation

$$\sigma^{-2}\int_0^\infty m(s)\,(1 - e^{-\sigma s})\,ds = \rho d^2(\lambda_n^2 + \alpha^2)^{-1}, \tag{3.1}$$

where the λ_n are again solutions of (1.3) or (1.4).

For a fluid with gradually fading memory, the material function $m(s)$ may be assumed to be negative for all values of s, and $|m(s)|$ may be considered to decrease monotonically to zero as s increases. If σ is real and positive, it follows that the left-hand side of (3.1) is negative. Since the right-hand side of (3.1) is always positive, it is clear that no real, positive value of σ can be a solution: that is to say, there can exist no non-oscillatory amplified disturbance.

If σ is complex, with real and imaginary parts σ_r and σ_i, the real and imaginary parts of (3.1) yield

$$(\sigma_r^2 - \sigma_i^2)\,I_1 + 2\sigma_r\sigma_i I_2 = \rho d^2(\sigma_r^2 + \sigma_i^2)^2\,(\lambda_n^2 + \alpha^2)^{-1},$$

$$2\sigma_r\sigma_i I_1 - (\sigma_r^2 - \sigma_i^2)\,I_2 = 0,$$

where $$I_1 = \int_0^\infty m(s)\,(1 - e^{-\sigma_r s}\cos\sigma_i s)\,ds, \quad I_2 = \int_0^\infty m(s)\,e^{-\sigma_r s}\sin\sigma_i s\,ds. \tag{3.2}$$

These equations lead to

$$I_1 = \frac{\rho d^2(\sigma_r^2 - \sigma_i^2)}{\lambda_n^2 + \alpha^2}, \quad I_2 = \frac{2\rho d^2\sigma_r\sigma_i}{\lambda_n^2 + \alpha^2}. \tag{3.3}$$

Now, let us suppose that there exists an amplified disturbance, for which

$\sigma_r > 0$. It is readily seen from (3.2) that both I_1 and $\sigma_i I_2$ must be negative for this disturbance. But (3.3) reveals that $\sigma_i I_2$ must have the same sign as σ_r. This contradiction proves that no amplified disturbance can exist.

The integrals I_1 and I_2 also furnish an upper limit for the damping rate of stable disturbances: for, in order that these integrals should remain bounded when $\sigma_r < 0, \exp(\sigma_r s)$ cannot tend to zero faster than $m(s)$. Therefore, if $m(s) \sim \exp(-s/\tau)$ for large values of s, where τ is some constant, the maximum damping rate must satisfy the inequality $\max|\sigma_r| < \tau^{-1}$.

4. Solution for $m(s) = -Ke^{-s/\tau}$

When the material function $m(s)$ has the particular form

$$m(s) = -Ke^{-s/\tau} \quad (s \geqslant 0),$$

where K and τ are positive constants, equation (3.1) has the solutions

$$\sigma = (2\tau)^{-1}[-1 \pm \surd(1-B)], \quad B = 4\tau^3(\lambda_n^2+\alpha^2)K/\rho d^2.$$

If $B < 1$, the two roots correspond to non-oscillatory, exponentially damped disturbances; while, if $B > 1$, complex conjugate roots occur, which represent damped oscillatory disturbances. The greatest- and least-damped modes have, respectively,

$$\begin{aligned} {\max \atop \min} |\sigma_r| &= (2\tau)^{-1}[+1 \pm \surd(1-B_0)] \quad \text{if} \quad B_0 < 1, \\ &= (2\tau)^{-1} \quad \text{if} \quad B_0 \geqslant 1, \end{aligned}$$

where B_0 is the minimum permissible value of B. Clearly, if $B_0 \geqslant 1$, *all* modes decay as $\exp(-t/2\tau)$.

We may relate τ, K and B to the material constants ρ, η_0 and γ. From results (2.2a, b) we have

$$\tau = -\gamma/\eta_0, \quad K = \eta_0^3/\gamma^2, \quad B = -4(\lambda_n^2+\alpha^2)\gamma/\rho d^2.$$

Now, the minimum value of $\lambda_n^2+\alpha^2$ is found to be 9·28, with $\alpha = 1{\cdot}2$ and λ_n equal to the lowest root of equation (1.3). Thus, $B_0 = -37{\cdot}1\gamma/\rho d^2$, which is greater or less than unity according as d^2 is less or greater than $37{\cdot}1|\gamma/\rho|$: i.e. for sufficiently thin layers, all modes decay as $\exp(-t\eta_0/|2\gamma|)$, but, for thicker layers, some non-oscillatory disturbances may decay less rapidly.

The result for a Newtonian fluid may be recovered on letting γ, and hence B, tend to zero. Then, the two roots for σ are $-\infty$ and $-(\lambda_n^2+\alpha^2)\eta_0/\rho d^2$, the latter of which is given by Yih (1963).

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